ADVANCED EXTENSION 1 TOPICS

MATHEMATICAL INDUCTION

1. If $u_1 = 7$ and $u_n = 2u_{n-1} - 1$ for $n \geq 2$, show by induction that $u_n = 3 \times 2^n + 1$ for $n \geq 1$.

2. If $u_1 = 5$, $u_2 = 11$ and $u_n = 4u_{n-1} - 3u_{n-2}$ for $n \geq 3$, show by induction that $u_n = 2 + 3^n$ for $n \geq 1$.

3. If $u_1 = 8$, $u_2 = 20$ and $u_n = 4u_{n-1} - 4u_{n-2}$ for $n \geq 3$, show by induction that $u_n = (n + 3)2^n$ for $n \geq 1$.

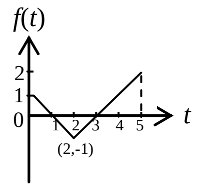
4. If $u_1 = 7$, $u_2 = 29$ and $u_n = 7u_{n-1} - 10u_{n-2}$ for $n \geq 3$, show by induction that $u_n = 2^n + 5^n$ for $n \geq 1$.

5. If $u_1 = 5$ and $u_n = 3u_{n-1} + 2$ for $n \geq 2$, show by induction that $u_n = 2 \times 2^n - 1$ for $n \geq 1$.

PROPERTIES OF INTEGRALS

1. Find the integral of the function $f(x)$ on the stated interval
   (a) between 0 and 5 if $f(x) = x + 2$ if $0 \leq x \leq 2$
   $= 4$ if $2 < x \leq 5$
   (b) between 0 and 2 if $f(x) = |x - 1|$
   (c) between $-\pi$ and π if $f(x) = |\sin x|$
   (d) between $-\pi$ and π if $f(x) = \cos x$ if $-\pi \leq x \leq 0$
   $= x$ if $0 < x < \pi$

2. The graph of the function
   $f(t) = 1 - t$, for $0 \leq t \leq 2$
   $= t - 3$, for $2 < t \leq 5$
   is shown in the diagram.
   The function $h(x)$ is defined as $\int_0^x f(t)dt$, $0 \leq x \leq 5$



   (a) Find the function $h(x)$ over the intervals $0 \leq x \leq 2$ and $2 < x \leq 5$
   (b) Find all the turning points of the function $h(x)$ and hence sketch it
   (c) Find $\int_0^5 f(x)dx$
   (d) Find the area between the curve $y = h(x)$, the x axis for $0 \leq x \leq 5$

3. Sketch the function $f(x)$ and prove the following inequalities
   (a) $f(x) = \frac{1}{x}$ $\quad 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} \geq \log_e n$
   (b) $f(x) = \frac{1}{\sqrt{x}}$ $\quad \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} \leq 2(\sqrt{n} - 1)$
   (c) $f(x) = \sqrt{x}$ $\quad \sqrt{1} + \sqrt{2} + ... + \sqrt{n} \geq \int_0^n x\,dx = \frac{2}{3}n\sqrt{n}$
   (d) $f(x) = x^2$ $\quad 1^2 + 2^2 + ... + (n-1)^2 \leq \frac{n^3}{3} \leq 1^2 + 2^2 + ... + n^2$

POLYNOMIALS

1. Express cos 5θ as a polynomial in cos θ, and obtain a similar expression for sin 5θ as a polynomial in sin θ.
   (a) Solve the equation cos 5θ = 1 for 0 < θ 2π and hence find the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$. Hence prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$ and $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$.
   (b) Solve the equation cos 5θ = 0 for 0 < θ 2π and hence find the roots of the equation $16x^4 - 20x^5 + 5 = 0$. Determine the exact value for $\cos\frac{\pi}{10}\cos\frac{3\pi}{10}$ and $\cos^2\frac{\pi}{10} - \cos^2\frac{3\pi}{10}$.
   (c) Solve the equation sin 5θ = 1 for 0 < θ 2π and hence find the roots of the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$. Determine the exact value of $\sin\frac{\pi}{10}\sin\frac{3\pi}{10}$.
   (d) Show that the roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \pm\sin\frac{\pi}{5}\sin\frac{2\pi}{5}$.and prove that $\sin^2\frac{\pi}{5}\sin^2\frac{2\pi}{5} = \frac{5}{4}$.

2. Prove that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta - 7\cos\theta$.
   (a) Hence find the roots of the equation $64x^6 - 112x^4 + 56x^2 - 7 = 0$. Deduce that $\cos\frac{\pi}{14} . \cos\frac{3\pi}{14} . \cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$ and $\cos^2\frac{\pi}{14} + \cos^2\frac{3\pi}{14} + \cos^2\frac{5\pi}{14} = \frac{7}{8}$.
   (b) Solve the equation cos 7θ = 1 for 0 < θ 2π, and hence find the roots of the equation $64x^7 - 112x^5 + 56x^3 - 7 = 1$. Deduce that $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$ and $\cos\frac{\pi}{7} . \cos\frac{\pi}{7} . \cos\frac{5\pi}{7} = -\frac{5}{4}$.