COMPLEX NUMBERS

EXERCISE 1

1. Solve for z, expressing answers in the form $a + ib$.
   (a) $(1 + i)z = 2 - i$
   (b) $\frac{2z}{2+i} + 3 - 2i = (1 - i)z$
   (c) $\frac{2}{z} = 1 + i + \frac{3}{1-i}$
   (d) $\frac{z+3}{z-1} = 2 - 3i$

2. Find the quadratic equation with roots
   (a) $i, -i$ (b) $1 + i, 1 - i$ (c) $2 + 3i, 2 - 3i$ (d) $3 + i, 1 + 3i$
   (e) $2 + i, \frac{1}{2+i}$

3. Solve for z and w
   (a) $z + iw = 2 + 3i$
   $z - iw = 2 - 3i$
   (b) $2x + w = 1 + i$
   $z - w = 1 - i$
   (c) $(2 + i)z + (2 - i)w = 1$
   $(2 - i)z + (2 + i)w = 2$
   (d) $z + (1 - i)w = 2i$
   $w + (1 - i)z = 1$

4. Show that $x = i$ is a root of the equation $x^3 + (1 - i)x^2 + (1 - 2i)x = 1 + i$

5. If $x = 1 + i$ is a root of $x^3 + ax + 4 = 0$, show that $a = -2$

6. If $\sqrt{x} + iy = a + ib$ where x, y, a, b are real and $a > 0$, provate that $a^2 - b^2 = x$ and $2ab = y$.
   Hence express the square root of the following in the form $a + ib$
   (a) $5 + 2i$ (b) $21 - 20i$ (c) i (d) $-11 - 60i$

EXERCISE 2

1. If ω is a complex cube root of unity (ie a root of $z^3 = 1$), prove that ω^2 is also a complex cube root of unity. Further prove that:
   (a) $1 + \omega + \omega^2 = 0$
   (b) $\frac{1}{1+\omega} + \frac{1}{1+\omega^2} = 1$
   (c) $(1 + \omega)^3 = -1$
   (d) $(1 + \omega^2)^5 = -\omega^2$

2. ω is a complex root of the equation $z^3 - 1 = 0$. Form a quadratic equation whose roots are given by $\alpha = 2 + \omega$ and $\beta = 2 + \omega^2$.

3. If ω is the complex cube root of unity, show that
   (a) $(1 + \omega - \omega^2) - (1 - \omega + \omega^2)^3 = 0$
   (b) $\frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} = \omega^2$
   (c) $\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = \omega$

4. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, where 1, ω, ω^2 are the other roots of unity, prove that
   (a) $x + y + z = 0$
   (b) $(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^2 + b^2 + c^2 - ab - bc - ca$

5. If 1, ω, ω^2 are the three cube roots of unity, prove that
   $(a + b + c)(a - b\omega + c\omega^2)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = a^3 + b^3 + c^3 - 3abc$

6. If ω is complex root of $z^5 - 1 = 0$, show that ω^2, ω^3, ω^4 are the other complex roots.
   (a) Prove that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$
   (b) Find the quadratic equation whose roots are $\alpha = \omega + \omega^4$ and $\beta = \omega^2 + \omega^3$
   (c) Show the roots of $z^5 - 1 = 0$ on an Argand diagram
   (d) Find the area of the pentagon formed by the roots (to 2 dec. Pl)

7. If ω is a complex root of $z^6 - 1 = 0$ then show that the other roots are ω^2, ω^3, ω^4, ω^5. Prove that
   (a) $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$
   (b) Find all the roots in the form $a + ib$ and indicate these roots in an Argand diagram.
   Find the area of the hexagon formed by the roots.
   (c) Find the quadratic equation whose roots are
   (i) ω and ω^5 (ii) ω^2 and ω^4
   (d) Show that
   (i) $z^6 - 1 = (z^2 - 1)(z^2 + z + 1)(z^2 - z + 1)$
   $= (z - 1)(z + 1)(z - \omega)(z - \omega^5)(z - \omega^2)(z - \omega^4)$
   (ii) The roots of $z^4 + z^2 + 1 = 0$ are ω, ω^2, ω^4 and ω^5

<u>EXERCISE 3</u>

1. If $z_1 = 2 + 3i$, $z_2 = -1 + 4i$, show on separate Argand diagrams
   (a) z_1
   (b) z_2
   (c) $z_1 + z_2$
   (d) $z_1 - z_2$
   (e) $z_2 - z_1$
   (f) $z_1 z_2$
   (g) iz_1
   (h) iz_2

2. Show on separate Argand diagrams the points representing
   (a) $2 - i$
   (b) $3 + 4i$
   (c) $(2 - i) + (3 + 4i)$
   (d) $(2 - i) - (3 + 4i)$
   (e) $(2 - i)(3 + 4i)$
   (f) $i(2 - i)$
   (g) $i(3 + 4i)$

3. Verify the triangle inequalities, $|z_1 + z_2| \leq |z_1| + |z_2|$ and $|z_1 - z_2| \geq |z_1| - |z_2|$ when
   (a) $z_1 = 2 + 3i$, $z_2 = -1 + 4i$
   (b) $z_1 = 2 - i$, $z_2 = 3 + 4i$

<u>EXERCISE 4</u>

1. If P represents the complex number z, sketch the locus of P if
   - (a) $|z| = 4$
   - (b) $|z| \leq 4$
   - (c) $|z - 3| < 3$
   - (d) $|z + 3i| < 1$
   - (e) $|2z - 3| = 1$
   - (f) $|z - 1 - 2i| = 4$
   - (g) $\arg z = \frac{\pi}{2}$
   - (h) $\arg z = -\frac{\pi}{3}$
   - (i) $\text{Re}(z) = 2$
   - (j) $\text{Im}(z) = -2$
   - (k) $1 < |z| < 2$
   - (l) $3 < |z| \leq 4$
   - (m) $2 \leq |z| \leq 5$
   - (n) $1 \leq |z + 2| \leq 2$
   - (o) $2 \leq \text{Im}(z) < 3$
   - (p) $2 < \text{Re}(z) \leq 3$
   - (q) $0 < \arg z < \frac{\pi}{6}$
   - (r) $\frac{\pi}{2} < \arg z < \frac{2\pi}{3}$
   - (s) $\left|\frac{\pi}{2}\right| \geq \frac{1}{9}$
   - (t) $1 < |z - 1 + i| < 2$
   - (u) $\text{Re}(z^2) = 0$
   - (v) $\text{Im}(z^2) = 2$
   - (w) $\text{Re}(z) = |z - 1|$
   - (x) $0 < \text{Re}(z) \leq 2$
   - (y) $\text{Re}(z - iz) \geq 2$

2. Mark clearly on an Argand diagram the regions of the z plane satisfied by
   - (a) $\text{Re}(z) \geq 1$ and $1 \leq \text{Im}(z) \leq 2$
   - (b) $3 < |z|$ and $\frac{\pi}{4} < \arg z \leq \pi$
   - (c) $|z| \leq 3$ and $\text{Im}(z) > 1$
   - (d) $2 < |z| \leq 3$ and $\text{Im}(z) > 1$
   - (e) $\text{Im}(z) \geq 1$ and $0 \leq \arg z \leq \frac{\pi}{4}$
   - (f) $1 \leq \text{Re}(z) \leq 2$ and $2 \leq \text{Im}(z) \leq 3$
   - (g) $1 < |z + i| < 2$ and $\pi < \arg z < \frac{3\pi}{2}$
   - (h) $4 \leq \text{Im}(z) \leq 4$ and $|z| \geq 5$
   - (i) $|2z - 3| < 2$ and $\frac{\pi}{6} < \arg z < \frac{\pi}{2}$

EXERCISE 5

1. Find the Cartesian equation of the following curves, and sketch and describe them
   (a) $|z - 2| = |z + i|$
   (b) $|z + 2 - 3i| = |z + 2 + i|$
   (c) $|z - 2i| = 2|z + 1|$
   (d) $|z + 2 - 3i| = 2|z + 2 + i|$

2. For the following, describe the locus of the complex number w, where z is restricted as indicated
   (a) $w = z - 2$, $|z| = 3$
   (b) $w = \frac{z-2}{z}$, $|z| = 1$
   (c) $w = \frac{z-2i}{z}$, $|z| = 2$
   (d) $w = \frac{z-2+i}{z+2-i}$, $|z| = 1$

3. Find the locus of z if
   (a) $w = \frac{z-1}{z}$ and w is purely real
   (b) $w = \frac{z-i}{z-2}$ and w is purely imaginary
   (c) $w = \frac{z-2}{z+2}$ and $\arg w = \frac{\pi}{3}$

4. Sketch on an Argand diagram the locus of the point P representing z, given that $|z|^2 = z + \bar{z} + 1$.

5. $|z + i| \le 2$ and $0 \le \arg(z + 1) \le \frac{\pi}{4}$. Sketch the region in the Argand diagram which contains the point P representing z.

6. $|z - 1| \le |z - i|$ and $|z - 2 - 2i| \le 1$. Sketch the region in the Argand diagram which contains the point P representing z. If P describes the boundary of this region, find the value of z when $\arg(z - 1) = \frac{\pi}{4}$.

7. $|z - 1| = 1$. Sketch the locus of the point P representing z on an Argand diagram. Hence deduce that $\arg(z - 1) = \arg(z^2)$.

8. $\text{Arg}(z + 3) = \frac{\pi}{3}$. Sketch the locus of the point P representing z on an Argand diagram. Find the modulus and argument of z when $|z|$ takes its least value. Hence find, in the form $a + ib$, the value of z for which $|z|$ is a minimum.

9. $z = x + iy$ is such that $\frac{z-i}{z-1}$ is purely imaginary. Find the equation of the locus of the point P representing z and show this locus on an Argand diagram.

10. $\text{Re}\left(z - \frac{1}{z}\right) = 0$. Find the equation of the locus of the point P representing z on an Argand diagram and sketch this locus.

11. Find the locus of z if
   (a) $\left|\frac{z-i}{z+2}\right| = 1$
   (b) $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{2}$
   (c) $\left|\frac{z-2}{z+2}\right| \le 1$
   (d) $2(z + \bar{z}) - 5i(z - \bar{z}) = 21$
   (e) $z\bar{z} - (2 + i)z - (2 - i)\,\bar{z} \le 4$
   (f) $\arg\left(\frac{z-i}{z+2}\right)=0$
   (g) $|z + 3i|^2 + |z - 3i|^2 = 90$