DAPTO HIGH SCHOOL

YEAR 12

EXTENSION 1 MATEMATICS

HALF YEARLY EXAMINATION

2010

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

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Answer each question on a SEPARATE sheet of paper. Extra paper is available.

Question 1 (12 marks) Use a SEPARATE sheet of paper.

(a) Solve $\frac{3}{x-2} < 1$ **3**

(b) The interval *AB*, where *A* is (4, 5) and *B* is (19, –5), is divided internally In the ratio 2:3 by the point P(*x*, *y*). Find the values of *x* and *y*. **2**

(c) The graphs of the line $x - 2y + 3 = 0$ and the curve $y = x^3 + 1$ intersect at (1, 2)

(i) Find the gradient of the tangent to the curve $y = x^3 + 1$ at (1, 2). **1**

(ii) Hence, find the size of the acute angle, in degrees, between the line and the tangent to the curve at the point of intersection. **2**

(d) Evaluate $\lim_{x \to 0} \frac{3x}{\sin 2x}$ **2**

(e) Find $\int \frac{1}{x^2 + 49}\, dx$ **2**

Question 2 (12 marks) Use a SEPARATE sheet of paper.

(a) Use the substitution $u = x^2 + 2$ to evaluate $\int \frac{x}{(x^2+2)^2}\,dx$ **3**

(b) The polynomial $P(x) = x^3 + ax + 12$ has a factor of $(x + 3)$. Find the value of a. **2**

(c) (i) Express $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$ where $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve $3\sin x + 4\cos x = 5$ for $0 \leq x \leq 2\pi$. Give your Answer or answers as radians correct to two decimal places. **2**

(d) Use Mathematical Induction to prove that, for every positive integer n, 13×6^n is divisible by 5. **3**

Question 3 (12 marks) Use a SEPARATE sheet of paper.

(a) Use the trapezoidal rule with 5 function values to find an approximation to

$$\int_3^7 e^{2x}dx$$

(b) It is known that the equation $x + \ln x - 2 = 0$ has a root which is near $x = 1.5$.

Use Newton's method once to find a better approximation to this root.

Give your answer correct to 4 significant figures. **3**

(c) One the first day of the harvest, an orchard produces 560 kg of fruit. On the next day, the orchard produces 543 kg, and the amount produces continues to decreased by the same amount each day.

(i) How much fruit is produced on the fourteenth day of the harvest? **2**

(ii) What is the total amount of fruit that is produced in the first 14 days of the harvest? **1**

(iii) On what day does the daily production first fall below 60 kg? **2**

(d) If α, β and γ are the roots of the equation $x^3 + 4x^2 - 6x - 8 = 0$.

Find the value of:

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **1**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **1**

Question 4 (12 marks) Use a SEPARATE sheet of paper.

(a) (i) State the domain and range of $y = 3\cos^{-1}2x$. **2**

(ii) Differentiate $y = 3\cos^{-1}2x$. **2**

(b) The points $P\ (2ap, ap^2)$ and $Q\ (2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.
The equation of the normal to the parabola at *P* is $x + py = 2ap + ap^3$ and the Equation of the normal at *Q* is $x + qy = 2aq + aq^3$.

(i) Show that the normal at *P* and *Q* intersect at the point *R* whose coordinates are $[-apq(p + q), a(p^2 + pq + q^2 + 2)]$. **2**

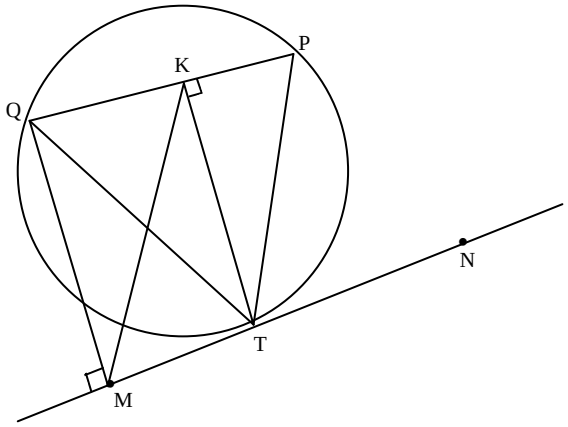
(ii) The equation of the chord *PQ* is $y = \frac{1}{2}(p + q)x - apq$. (Do NOT show this.)
If the chord *PQ* passes through the focus (0, a), show that $pq = -1$. **1**

(iii) Find the equation of the locus of *R* if the chord passes through (0, a). **2**

(c) Find $\int_0^{\frac{\pi}{4}} \sin^2 x\, dx$. **3**

Question 5 (12 marks) Use a SEPARATE sheet of paper.

(a)



The points *P*, *Q* and *T* lie on a circle. The line *MN* is a tangent to the circle at *T* with *M* chosen so that *QM* is perpendicular to *MN*. The point *K* on *PQ* is chosen so that *TK* is perpendicular to *PQ* as shown in the diagram.

(i) Show that *QKTM* is a cyclic quadrilateral. **1**

(ii) Show that $\angle KMT = \angle KQT$. **1**

(iii) Hence, or otherwise, show that *MK* is parallel to *TP.* **2**

(b) The curve $y = x^3$, between $y = 1$ and $y = 8$, is rotated about the *y*-axis.
Find the volume of the solid generated. **3**

(c) (i) Factorise fully $y = x^3 - 4x$. **1**

(ii) Sketch the polynomial $y = x^3 - 4x$. **2**

(iii) Calculate the area bounded by the curve $y = x^3 - 4x$, the *x*-axis and the lines $x = 1$ and $x = 2$. **2**

Question 6 (12 marks) Use a SEPARATE sheet of paper.

(a) If $y = 7e^{2x-3}$ show that $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$. **2**

(b) Find $\frac{d}{dx}\left(\frac{\ln x}{x}\right)$. **2**

(c) Use the table of standard integrals to evaluate $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$. **2**

(d) Evaluate $\log_7 3$ correct to four significant figures. **1**

(e) By making the substitution $t = \tan\frac{\theta}{2}$, or otherwise, show that
$\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}$. **2**

(f) (i) By expanding the left-hand side, show that
$\sin(5x + 4x) + \sin(5x - 4x) = 2\sin 5x \cos 4x$. **2**

(ii) Hence find $\int \sin 5x \cos 4x \, dx$ **1**

Question 7 (12 marks) Use a SEPARATE sheet of paper.

(a) Find $\int_0^1 \frac{dx}{2x+1}$. **2**

(b) Solve $2\log_{10} x + \log_{10} 8x = 3$. **2**

(c) If $f(x) = x^2 + 3x$, find $f'(x)$ from first principles. **2**

(d) A couple borrows \$70 000 and agrees to repay the principle plus interest in equal monthly instalments of \$$M$. Reducible interest of 0.65% per month is charged monthly on the amount owing.

(i) Show that the amount owing after 3 months, (\$$A_3$), is given by:
$A_3 = 70000(1.0065)^3 - M(1 + 1.0065 + 1.0065^2)$. **3**

(ii) Write down an expression for the amount owing after n months (A_n). **1**

(iii) Find the amount of each instalment, to the nearest dollar, if the loan is paid at the end of 10 years. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 1$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right) \quad x < a < 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$