DAPTO HIGH SCHOOL

YEAR 12

EXTENSION 2 MATEMATICS

HALF YEARLY EXAMINATION

2010

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of the paper
- All necessary working should be shown in every question

Total marks – 80
- Attempt Questions 1 – 4
- All questions are of equal value

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- **Attempt Questions 1 – 4**
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Answer each question on a SEPARATE sheet of paper. Extra paper is available.

Question 1 (20 marks) Use a SEPARATE sheet of paper.

(a) Given that $z = 2 - i$ and $w = 4 + 2i$, find,

(i) $iz - 3w$ **2**

(ii) $z\overline{w}$ **2**

(iii) $\frac{5}{z}$ **2**

(b) Find $\sqrt{3 - 4i}$ (i.e. if $z^2 = 3 + 4i$, find z where $z = x + iy$) **3**

(c) Sketch the locus of z satisfying the following:

(i) $\text{Im}(z) = -3$ **1**

(ii) $\text{Arg}(z - 4) = \frac{\pi}{3}$ **1**

(iii) $|z| \leq 4$ **1**

(e) (i) Write $\sqrt{3} + i$ in modulus argument form. **2**

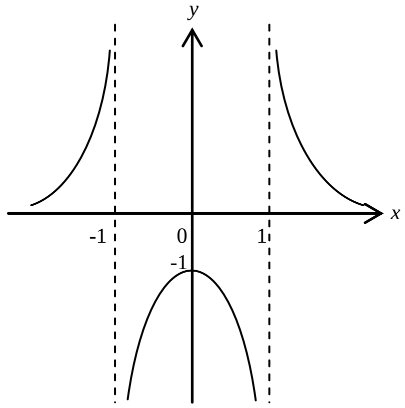
(ii) Hence, write $\left(\sqrt{3} + i\right)^6$ in simplest form. **3**

(f) The points B and C are represented by the complex numbers $z = 1 - 3i$ and $w = -3 + 4i$ respectively. Find the point on the positive real x-axis so that ΔBPC is right angled, with $\Delta BPC = 90°$. **3**

Question 2 (20 marks) Use a SEPARATE sheet of paper.

(a) The diagram on the right shows the graph of $y = f(x)$.

Draw sketches of each of the following:



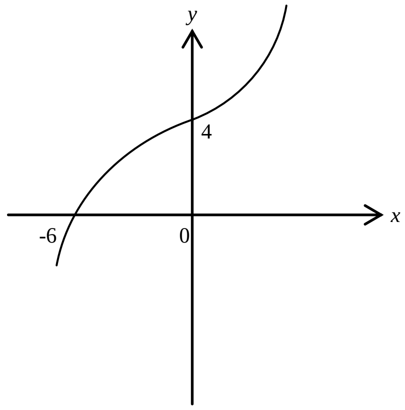
(i) $y = f(|x|)$

(ii) $y = \sqrt{f(x)}$

(iii) $y^2 = f(x)$

(b) The diagram on the right shows the graph of $y = f(x)$.

Draw sketches of each of the following:



(i) $y = |f(x)|$

(ii) $y = f(|x|)$

(iii) $y = \sqrt{f(x)}$

(c) The equation of a curve is given by $xy + x^2 = 1$.

(i) Use implicit differentiation to find $\frac{dy}{dx}$. **2**

(ii) Rewrite the equation of the curve to show that $y = \frac{1}{x} - x$. **2**

(iii) Write down the equations of the *two* asymptotes. **2**

(iv) What are the intercepts of this curve on the x-axis? **2**

(v) Sketch this curve. There are no stationary points or points of inflexion. **2**

(d) (i) Factorise fully $f(x) = x^4 - 4x^3 + 4x^2$. **2**

(ii) Sketch the graph of $f(x) = x^4 - 4x^3 + 4x^2$. Do NOT use calculus. **1**

(e) Given that $g(x) = f(x) - \frac{1}{2}$, sketch the graph of:

(α) $y = g(x)$ **1**

(β) $y = |g(x)|$ **1**

Question 3 (20 marks) Use a SEPARATE sheet of paper.

(a) $P(x_1, y_1)$ is a point on the ellipse with equation $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

(i) Calculate the eccentricity of the ellipse. **2**

(ii) If S and S' are the foci of the ellipse, find the coordinates of S and S' and write the equations of the corresponding directrices. **2**

(iii) Use implicit differentiation to find $\frac{dy}{dx}$ for the ellipse. **1**

(iv) Show that the equation of the tangent at *P* is given by $\frac{x_1 x}{25} + \frac{y_1 y}{16}$. **2**

(v) The tangent at *P* intersects the coordinate axes at *A* and *B*.

(α) Find the coordinates of A and B. **2**

(β) Show that the area of ΔAOB, where O is the origin, is given by $\frac{200}{x_1 y_1}$. **1**

(b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is the equation of a hyperbola.

(i) Find the equation of each asymptote tho this hyperbola. **2**

(ii) Show that, for this hyperbola, $\frac{dy}{dx} = \frac{9x}{16y}$. **1**

(iii) Hence, show that the gradient of the tangent to this hyperbola at the point $(4\sec\theta, 3\tan\theta)$ is $\frac{3}{4}\text{cosec}\theta$. **2**

(c) The tangent at the point $P(cp, \frac{c}{p})$ on the hyperbola $xy = c^2$ is given by the equation $x + p^2 y = 2cp$. This tangent cuts the asymptotes to the hyperbola at the points *A* and *B*.

(i) What are the coordinates of *A* and *B*? **2**

(ii) Prove that $AB = 2PO$, where *O* is the origin. **3**

Question 4 (20 marks) Use a SEPARATE sheet of paper.

(a) (i) Show, by long division, that $\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}$. **2**

(ii) Hence, find $\int \frac{x^2}{x^2+1}\,dx$ **2**

(b) Using a table of standard integrals, or otherwise, find $\int \frac{dx}{\sqrt{9x^2-1}}$ **2**

(c) Evaluate $\int_0^1 x^2 e^{-4x^3}\,dx$ by first making a suitable substitution. **3**

(d) Find $\int \cos x \sin^3 x\,dx$ **2**

(e) (i) Reduce $\frac{x+1}{(x-2)(x-3)}$ to its partial fractions. **2**

(ii) Hence evaluate $\int_4^5 \frac{x+1}{(x-2)(x-3)}\,dx$. Leave your answer in exact form. **3**

(f) By first making an appropriate trigonometric substitution, find $\int \frac{dx}{(1+x^2)^2}$ **4**

End of Paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 1$$

$$\int \frac{1}{x} dx = \mathrm{In}\, x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \mathrm{In}\left(x + \sqrt{x^2 - a^2}\right) \quad x < a < 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \mathrm{In}\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\mathrm{In} x = \log_e x, \quad x > 0$