Inverse Functions

- **Inverse Functions**
  - if $y = x^3$, then inverse is $x = y^3$ or $y = \sqrt[3]{x}$
  - this is found by interchanging the x and y positions, the solving for either variable (usually y)
  - hence the domain and range are also interchanged
  - the inverse function is donated by $y = f^{-1}(x)$
  - a function has an inverse if a horizontal line cuts the graph of the function at only one point (test)
  - by restricting the domain on certain functions, we are able to find their inverse functions, as they can then satisfy the horizontal line test

- **An important relationship**
  - $\frac{dy}{dx} . \frac{dx}{dy} = 1$

- **The Inverse Trigonometric Functions**
  Inverse Sine Function
  - for $y = \sin x$, inverse is $x = \sin y$
  - $y = \sin^{-1}x$
  - domain: $-1 \le x \le 1$
  - range: $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$
  - these are the opposite (inverse) of y = sinx
  - if $y = \sin^{-1}x$ it follows immediately that $x = \sin y$

  Inverse Cosine Function
  - for $y = \cos x$, inverse is $x = \cos y$
  - $y = \cos^{-1}x$
  - domain: $-1 \le x \le 1$
  - range: $0 \le y \le \pi$
  - these are the opposite (inverse) of $y = \cos x$
  - if $y = \cos^{-1}x$ it follows immediately that $x = \cos y$

  Inverse Tangent Function
  - for $y = \tan x$, inverse is $x = \tan y$
  - $y = \tan^{-1}x$
  - domain: all real x
  - range: $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$
  - these are the opposite (inverse) of $y = \tan x$
  - if $y = \cos^{-1}x$ it follows immediately that $x = \cos y$

  For functions such as y = $a\sin^{-1}bx$, eg $y = 3\sin^{-1}2x$:
  - the domain is found by multiplying the b value by standard domain x,
    eg $-1 \le \mathbf{2 \times x} \le 1$ so domain will be $\frac{-1}{2} \le x \le \frac{1}{2}$
  - the domain is found by multiplying the a value by the limits of standard range,
    eg $\mathbf{3 \times \frac{-\pi}{2}} \le y \le \mathbf{3 \times \frac{\pi}{2}}$ so range will be $\frac{-3\pi}{2} \le y \le 3 \times \frac{3\pi}{2}$

- **Properties of Inverse Functions**
  - $\sin^{-1}(-x) = -\sin^{-1}x$
  - $\cos^{-1}(-x) = \pi - \cos^{-1}x$
  - $\tan^{-1}(-x) = -\tan^{-1}x$

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$
- $\sin(\sin^{-1}x) = x$
- $\cos(\cos^{-1}x) = x$
- $\tan(\tan^{-1}x) = x$

- **General Solutions of Trigonometric Equations**
  - if $\sin\theta = b$, then $\theta = n\pi + (-1)^n \sin^{-1}b$
  - if $\cos\theta = b$, then $\theta = 2n\pi \pm \cos^{-1}$
  - if $\tan\theta = b$, then $\theta = n\pi + \tan^{-1}$

- **Derivative of Inverse Trigonometric Functions**
  - $\frac{d}{dx}\left[\sin^{-1}x\right] = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\left[\cos^{-1}x\right] = \frac{-1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\left[\tan^{-1}x\right] = \frac{1}{1+x^2}$
  - $\frac{d}{dx}\left[\sin^{-1}f(x)\right] = \frac{f'(x)}{\sqrt{1-[f'(x)]^2}}$
  - $\frac{d}{dx}\left[\cos^{-1}f(x)\right] = \frac{-f'(x)}{\sqrt{1-[f'(x)]^2}}$
  - $\frac{d}{dx}\left[\tan^{-1}f(x)\right] = \frac{f(x)}{1+[f(x)]^2}$
  - $\frac{d}{dx}\left[\sin^{-1}\frac{x}{a}\right] = \frac{1}{\sqrt{a^2+x^2}}$
  - $\frac{d}{dx}\left[\cos^{-1}\frac{x}{a}\right] = \frac{-1}{\sqrt{a^2-x^2}}$
  - $\frac{d}{dx}\left[\tan^{-1}\frac{x}{a}\right] = \frac{a}{a^2+x^2}$

- **Integrals Involving Inverse Trigonometric Ratios**
  - $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$ or $-\cos^{-1}x + C$
  - $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$ or $-\cos^{-1}\frac{x}{a} + C$
  - $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$
  - $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$