

SOLUTIONS

$$\begin{aligned} \textcircled{1} \quad x &= 4p \therefore p = \frac{x}{4} \\ y &= 2p^2 - 2 = 2 \left(\frac{x}{4} \right)^2 - 2 \\ &= \frac{2x^2}{16} - 2 \\ \therefore y &= \frac{x^2}{8} - 2 \end{aligned}$$

$$\textcircled{2} \quad (a) \quad x^2 = 20y \therefore 4a = 20$$

$$\therefore x = 10p, y = 5p^2$$

$$(b) \quad y = 4x^2 \therefore x^2 = \frac{y}{4}$$

$$\therefore 4a = \frac{1}{4}$$

$$a = \frac{1}{16}$$

$$\therefore x = \frac{1}{8}p, y = \frac{1}{16}p^2$$

$$\textcircled{3} \quad (a) \quad \text{gradient} = \frac{5q^2 - 5p^2}{10q - 10p} = \frac{5(q+p)(q-p)}{10(q-p)} = \frac{q+p}{2}$$

$$\text{Equation is } y - 5p^2 = \frac{q+p}{2}(x - 10p)$$

$$2y - 10p^2 = qx - 10pq + px - 10p^2$$

$$2y = (q+p)x - 10pq$$

$$y = \frac{(q+p)x}{2} - 5pq$$

(b) Focus is $(0, 5)$

$$\therefore 5 = \frac{(p+q)}{2}, 0 = 5pq$$

$$\therefore -5pq = 5$$

$$pq = -1$$

$$\textcircled{4} \quad (a) \quad y = \frac{x^2}{4a} \therefore \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}.$$

$$\therefore \text{at } P \quad \frac{dy}{dx} = \frac{2ap}{2a} = p$$

\therefore tangent is $y - ap^2 = p(x - 2ap)$

$$y - ap^2 = px - 2ap^2$$

$$y = px - ap^2$$

$$(b) \quad y = px - ap^2 \dots \textcircled{1}$$

$$y = qx - aq^2 \dots \textcircled{2}$$

$$\textcircled{2} - \textcircled{1} \quad qx - px - aq^2 + ap^2 = 0$$

$$x(q-p) = aq^2 - ap^2$$

$$x = \frac{a(q-p)(q+p)}{q-p} = a(q+p)$$

$$\text{Subs in } \textcircled{1} \quad y = pa(q+p) - ap^2 \\ = apq + ap^2 - ap^2 = apq$$

$$\therefore M \text{ is } [a(p+q), apq]$$

$$(c) \quad \left[\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2} \right]$$

$$R = \left[a(p+q), \frac{a(p^2+q^2)}{2} \right]$$

(d) Midpoint MR

$$= \left[a(p+q), \frac{apq + a(p^2+q^2)}{2} \right]$$

$$= \left[a(p+q), \frac{a(p+q)^2}{4} \right]$$

Now show this point lies on $x^2 = 4ay$

$$\begin{aligned} \text{LHS} &= x^2 \\ &= a^2(p+q)^2 \\ &= 4a \cdot a \frac{(p+q)^2}{4} \\ &= a^2(p+q)^2 \end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

\therefore parabola bisects MR

(5) S is the focus. $\therefore S$ is $(0, a)$

Tangent at P is $y = px - ap^2$

Equation of latus rectum is $y = a$

PQ is a focal chord $\therefore pq = -1$

To find L solve $y = a \dots \textcircled{1}$

$$y = pac - ap^2 \dots \textcircled{2}$$

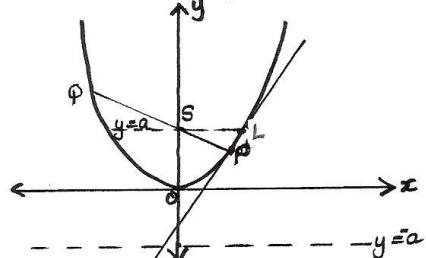
$$\therefore a = pac - ap^2$$

$$pac = ap^2 + a$$

$$\therefore x = ap + \frac{a}{p} = a(p + \frac{1}{p})$$

$$\therefore L \text{ is } \left[a(p + \frac{1}{p}), a \right]$$

Hence $SL = a(p + \frac{1}{p})$ (horizontal line)



Now $PS = \text{distance of } P \text{ from focus}$

$= \text{distance of } P \text{ from directrix}$

$$= ap^2 + a$$

And $PS = aq^2 + a$ (but $q = -\frac{1}{p}$)

$$= a\left(\frac{-1}{p}\right)^2 + a$$

$$= a\left(\frac{1}{p^2} + 1\right)$$

$$\begin{aligned} \text{Hence } PS \cdot PQ &= (ap^2 + a) \left(\frac{a}{p^2} + a \right) \\ &= a^2(p^2 + 1) \left(\frac{1}{p^2} + 1 \right) \\ &= a^2 \left(1 + p^2 + \frac{1}{p^2} + 1 \right) \\ &= a^2 \left(p^2 + \frac{1}{p^2} + 2 \right) \\ &= a^2 \left(p + \frac{1}{p} \right)^2 \end{aligned}$$

$$\text{And } SL^2 = [a(p + \frac{1}{p})]^2 = a^2(p + \frac{1}{p})^2$$

$$\therefore SL^2 = PS \cdot PQ$$

$$(6)(a) \quad y = \frac{2x^3}{8} \therefore y' = \frac{2x^2}{8} = \frac{2x}{4}$$

$$\text{at } P, y' = \frac{4p}{4} = p$$

$$\therefore \text{normal is } y - 2p^2 = -\frac{1}{p}(x - 4p)$$

$$py - 2p^3 = -x + 4p$$

$$x + py = 2p^3 + 4p$$

$$(b) \quad x + py = 2p^3 + 4p \dots \dots \textcircled{1}$$

$$x + qy = 2q^3 + 4q \dots \dots \textcircled{2}$$

$$\textcircled{1} - \textcircled{2} \quad py - qy = 2p^3 - 2q^3 + 4p - 4q$$

$$y(p-q) = 2(p-q)(p^2 + pq + q^2) + 4(p-q)$$

$$y = 2(p^2 + pq + q^2) + 4$$

$$= 2(p^2 + pq + q^2 + 2)$$

$$\text{subs in } \textcircled{1} \quad x + 2p(p^2 + pq + q^2 + 2) = 2p^3 + 4p$$

$$x + 2p^3 + 2p^2q + 2pq^2 + 4p = 2p^3 + 4p$$

$$x = -2p^2q - 2pq^2 = -2pq(p+q)$$

$$\therefore R \text{ is } [-2pq(p+q), 2(p^2 + pq + q^2 + 2)]$$

$$\text{But } pq = -1$$

$$\therefore R \text{ is } [-2(-1)(p+q), 2(p^2 - 1 + q^2 + 2)]$$

$$= [2(p+q), 2(p^2 + q^2 + 1)]$$

$$\therefore x = 2(p+q) \therefore p+q = \frac{x}{2} \dots \textcircled{3}$$

$$y = 2(p^2 + q^2 + 1)$$

$$= 2[(p+q)^2 - 2pq + 1]$$

$$= 2[(p+q)^2 + 3] \dots \dots \textcircled{4}$$

$$\text{subs } \textcircled{3} \text{ into } \textcircled{4} \quad y = 2 \left[\left(\frac{x}{2} \right)^2 + 3 \right]$$

$$= 2 \left[\frac{x^2}{4} + 3 \right]$$

$$= \frac{x^2}{2} + 6$$

$$\therefore x^2 = 2y - 12 = 2(y - 6)$$

$$(7)(a) \quad y = px - ap^2$$

(b) gradient of tangent is p
 \therefore gradient of normal is $-1/p$

\therefore equation of line from S is

$$y - a = -\frac{1}{p}(x - a)$$

$$py - pa = -x$$

$$x + py = pa$$

$$(c) \quad y = px - ap^2 \dots \dots \textcircled{1}$$

$$x + py = pa \dots \dots \textcircled{2}$$

subs $\textcircled{1}$ into $\textcircled{2}$ gives

$$x + p(px - ap^2) = pa$$

$$x + p^2x - ap^3 = ap$$

$$x(1 + p^2) = ap(p^2 + 1)$$

$$\therefore x = ap$$

$$\text{subs into } \textcircled{1} \quad y = p(ap) - ap^2$$

$$= ap^2 - ap^2$$

$$= 0$$

$$\therefore N \text{ is } (ap, 0)$$

Hence, the locus of N is $y = 0$

$$(8) \quad x^2 = 8y \therefore 4a = 8$$

\therefore Equation is $x x_0 = 2a(y + y_0)$

$$\text{i.e. } -2x = 2 \cdot 2(y - 4)$$

$$-2x = 4y - 16$$

$$2x + 4y - 16 = 0$$

$$x + 2y - 8 = 0$$

$$(9)(a) \quad y = tx - at^2$$

$$(b) \text{ when } y = 0, \quad 0 = tx - at^2$$

$$tx = at^2$$

$$x = at$$

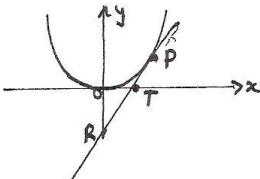
$$\therefore T \text{ is } (at, 0)$$

$$\text{when } x = 0, \quad y = t(0) - at^2$$

$$= -at^2$$

$$\therefore R \text{ is } (0, -at^2)$$

(c)



$$\begin{aligned} PT &= \sqrt{(2at - at)^2 + (at^2 - 0)^2} \\ &= \sqrt{(at)^2 + (at^2)^2} \\ &= \sqrt{a^2t^2 + a^2t^4} = at\sqrt{1+t^2} \end{aligned}$$

$$\begin{aligned}
 PR &= \sqrt{(2at - 0)^2 + (at^2 + at^2)^2} \\
 &= \sqrt{4a^2t^2 + 4a^2t^4} \\
 &= 2at\sqrt{1+t^2} \\
 \therefore \text{ratio is } at\sqrt{1+t^2} : 2at\sqrt{1+t^2} \\
 &= 1:2
 \end{aligned}$$

But since division is external then
the ratio is $-1:2$.

$$\begin{aligned}
 (d) SP &= \text{distance of } P \text{ from directrix} \\
 &= at^2 + a \\
 &= a(t^2 + 1)
 \end{aligned}$$

$$\begin{aligned}
 SR &= a + at^2 \\
 &= a(1 + t^2)
 \end{aligned}$$

$$\begin{aligned}
 \therefore SP &= SR \\
 \therefore \angle SPR &= \angle SRP \quad (\text{opp eq. sides in isos. } \Delta)
 \end{aligned}$$

$\therefore m$ makes equal angles with the lines
 SP and the y -axis.

$$\textcircled{10} \quad x^2 = 2y \quad \therefore 4a = 2, a = \frac{1}{2}$$

$$\therefore P \text{ is } \left(2 \cdot \frac{1}{2} \cdot p, \frac{1}{2} p^2\right) = (p, \frac{1}{2} p^2)$$

$$\text{midpoint of } OP = \left(\frac{0+p}{2}, \frac{0+\frac{1}{2}p^2}{2}\right)$$

$$= \left(\frac{p}{2}, \frac{p^2}{4}\right)$$

$$\therefore x = \frac{p}{2} \quad \text{and} \quad p = 2x$$

$$\text{subs into } y = \frac{p^2}{4} = \frac{(2x)^2}{4}$$

$$\therefore y = x^2$$