VOLUMES BY SLICING

EXERCISE 1

1. Find the volume of the solid generated by rotating about the x axis and the regions described below. Use slices perpendicular to the x axis.
   (a) $y = x^2$, $y = x$
   (b) $y = x^3$, $x = 2$ and the x axis
   (c) $y = x^{2/3}$, $x = 8$, $x = -8$ and the x axis
   (d) $x^{1/2} + y^{1/2} = a^{1/2}$, the x and y axes
   (e) $y = 4 - x^2$, $x = 2$ and $y = 4$

2. Find the volume of the solid generated by rotating about the y axis and the regions described below. Use slices perpendicular to the y axis.
   (a) $y = 4 - x^2$, $x = 2$ and $y = 4$
   (b) $y = \sqrt{x}$, $y = 2$ and the y axis
   (c) $y = \sqrt{x}$, $x = 4$ and the x axis
   (d) $y^2 = \frac{1}{2}x^3$, $x = 2$
   (e) $y = (x - 3)^3$, $y = 8$ and the coordinate axes

3. Find the volume of the solid of revolution obtained by rotating about the x axis, the region bounded by the curve $y = \sin x$ for $0 \leq x \leq \pi$ and the x axis. Use slices perpendicular to the x axis.

4. Find the volume of the sold of revolution when the area enclosed between the curve $x^2 = 4 - y$ and the lines $y = 4$ and $x = 2$ is rotated:
   (a) about the line $y = 4$ using slices perpendicular to the axis of revolution
   (b) about the line $x = 4$ using slices perpendicular to the axis of revolution

5. From an extremity A of the latus rectum AB of the parabola $x^2 = 4ay$, interval AK is drawn perpendicular to the x axis. Show that the volume formed by the rotation of the region OAK about the line AK is $\frac{2}{3}\pi a^3$ units^3.

6. The parabola $y^2 = 4ax$ is rotated about it latus rectum. Using slices perpendicular to the axis of rotation, find the volume of the solid generated.

7. Find the volume of the solid (called the prolate spheroid) by rotating the region enclosed by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ about the x axis. Use slices perpendicular to the x axis.

8. Find the volume of the solid (called the oblate spheroid) by rotating the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the y axis. Use slices perpendicular to the y axis.

9. Sketch the region in the Cartesian plane satisfying the conditions:
   $$0 \leq x \leq 2,\ 0 \leq y \leq x^2 - \tfrac{1}{4}x^4$$
   Rotate this region about the y axis and using slices perpendicular to the y axis, calculate the volume of the solid of revolution generated. (Hint: use calculus to find the limits of the definite integral.)

10. Find the volume of the solid of revolution obtained when the area cut off by the y axis, the line $y = e^2$ and the curve $y = e^{x}$ is rotated about the y axis. Use slices perpendicular to the y axis.

11. The region between the curve $x = 4y^2 + 2$ and the vertical line $x = 6$ is rotated about the y axis. Find the volume of the solid generated by slicing perpendicular to the y axis.

12. Find the volume generated when the area between the curve $y = \frac{1}{2}\sqrt{x - 2}$, the line $y = 1$ and the coordinate axes is rotated about the y axis. Use slices perpendicular to the axis of rotation.

13. A hole of radius y is bored through a sphere of radius R. Find the remaining volume if:
 (a) R = 8cm and r = 2cm
 (b) The radius of the sphere is 12 cm and the diameter of the hole is 6cm
 (c) $R : r = 2 : 1$
 (d) $R : r = 4 : 1$

14. Find the volume of the solid of revolution (called the TORUS) obtained when the circle $x^2 + (y - b)^2 = a^2$ $(b > a)$ is rotated about the x axis. Use slices perpendicular to the x axis.

15. Let n be a positive integer and a be a positive real number. Consider the graph of the function $y = x^n$ which divides the rectangle formed by the lines $x = a$, $y = a^n$, $x = 0$, $y = 0$ into two regions bounded by two straight lines and a section of the curve itself.
 (a) Show that the two solids obtained by rotating these regions about the x axis are in the ratio $1 : 2n$
 (b) Find the volume of the solid obtained by rotating the region formed by the curve $y = x^3$, $y = 8$, and the y axis. Compare this answer to the answer of Question 1(b) and use this to verify the result above.

EXERCISE 2

1. Calculate the volume of the solid generated when the region bounded by the curve $y = \sqrt{x}$, the y axis and the line $y = 1$ is rotated about the y axis. Use slices parallel to the y axis.

2. The region formed by the curve $y = 4 - x^2$, the lines $x = 2$ and $y = 4$ is revolved around the y axis. Use slices parallel to the y axis to find the volume of the solid of revolution.

3. Use cylindrical shells to find the volume of the solid obtained by rotating about the y axis, the region R where R = $\{(x, y)$: $0 \le x \le \pi$, $0 \le y \le \frac{\sin x}{x}\}$.

4. Find the volume of the torus (donut shape) formed when the circle $(x - b)^2 + y^2 = a^2$ (where $a < b$) is rotated about the y axis, using slices parallel to the y axis.
(Hint: substitute $x = b + a\cos\theta$) when integrating.)

5. The region defined by $0 \le x \le 2$, $0 \le y \le x^2 - \frac{1}{4}x^4$ is rotated about the y axis. Use shells to calculate the volume of the solid of revolution generated.

6. The region between the curve $y = \frac{1}{2}\sqrt{x-2}$, the line $x = 6$ and the x axis is revolved about the y axis. By using slices parallel to the y axis, find the volume of the solid generated.

7. Find the volume of the solid of revolution when the area enclosed between the curve $x^2 = 4 - y$ and the lines $y = 4$ and $x = 2$ is rotated:
   (a) about the line $y = 4$ using slices parallel to the line $y = 4$
   (b) about the line $x = 2$ using slices parallel to the line $x = 2$

8. Find the volume of the solid of revolution formed when the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is rotated through one complete revolution about the line $y = b$.

9. The area common to the curves $y = x^2$ and $y^2 = x$ is revolved about the y axis. Find the volume generated by using cylindrical shells.

10. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the line $x = a$. Find the volume generated by using the shell method.

11. Find the volume of the torus obtained by rotating the area bounded by the circle $x^2 + y^2 = a^2$ about the line $x = c$, $(c > a)$ using cylindrical shells.

12. The area bounded by the parabola $y^2 = 4ax$ and the line $x = a$ is rotated about the y axis. Find the volume by two methods.

13. The area enclosed by the ellipse $16x^2 + 25y^2 = 400$ is rotated about the line $x = 8$. By using the method of shells find the volume generated.

14. The region bounded by the curves $y = 3x - x^2$ and $y = x$ is rotated about the y axis. Find the volume by the shell method.

EXERCISE 3

1. The base of a certain solid is a circular disc, with radius 3cm. A cross-section perpendicular to the base and to a diameter of the disc is:
   (a) an isosceles triangle of height k with its base being in the plane of the circle
   (b) an equilateral triangle with one of its sides in the base of the solid
   (c) an isosceles right triangle with
      (i) the unequal side in the base of the solid
      (ii) one of the equal sides in the base of the solid
   (d) a square, with one side in the base of the solid
   (e) a semicircle with a diameter in the base of the solid
   Find the volume of these solids.'

2. The base of a certain solid is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Every cross-section perpendicular to the base and the major axis is:
   (a) a square with one side in the base of the solid
   (b) an isosceles right triangle with the base in the plane of the ellipse
   (c) a semicircle with its diameter in the base of the solid
   Find the volumes of these solids.

3. Find the volume of such a solid which is constructed with a square base of side s. Cross-sections perpendicular to the base and to one of its diagonals are squares also, with one of its sides in the base of the solid.

4. A solid in the shape of a wedge that is cut off a circular cylinder by a plane passing through the diameter of the base that is inclined to the base at an angle of θ has the radius of its base R. Show by using slices perpendicular to the line of intersection of the planes, that the volume is $\frac{2}{3}E^3\tan\theta$.

5. Calculate the volume of a solid ellipsoid such that three mutually perpendicular cross-sections are bounded by ellipse whose axes are $2a$, $2b$ and $2c$. The area of an ellipse is πab units^2.

6. A solid has its base the region bounded by the curves $y = x$ and $x = 2y - \frac{1}{2}y^2$. Cross-sections perpendicular to the base and to the y axis are:
   (a) semicircles with diameter the base
   (b) equilateral triangles with a side in the base.
   Find the volume of these solids.

7. The base of a certain solid is the region between the curves $y = x$ and $y = x^2$. Each plane section of the solid perpendicular to the base and to the x axis is a semicircle with its diameter in the base of the solid. Show that the volume of the solid is $\frac{\pi}{240}$ units^3.

8. The base of a solid is the parabolic segment of the parabola $y = x^2$, cut off by the chord $y = 4$. Each plane section of the solid perpendicular to the base and to the axis of the parabola is a rectangle, whose base is the chord and its height is $\frac{1}{2}(4 - y)$ units. Find the volume of this solid.